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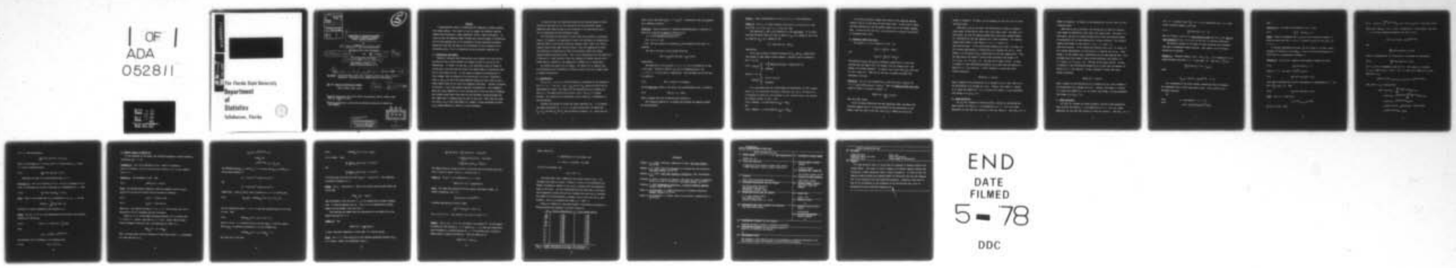

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A COMPARISON OF TWO SAMPLING SCHEMES
WHEN TESTING A SIMPLE HYPOTHESIS
VERSUS A SIMPLE ALTERNATIVE.

by

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Abstract

A large deviation result is established for sequences of random variables with random indices. This result is used to compare two different sampling schemes when testing for a simple hypothesis versus a simple alternative. It turns out that the sampling scheme yielding more expected number of observations for the same sampling cost is not necessarily the more profitable procedure. Properties other than the mean of the distribution of the occurrence of the observations play a role in determining the more profitable sampling rule.

1. Introduction and summary.

Consider a situation where observations occur randomly over time and the statistician has to choose between two schemes by which he can pay for the observations. In scheme I he must pay c_1 per observation while in scheme II he must pay c_2 per unit time and collect the random number of observations that fall in that period of time. In this paper we compare the profitability of these schemes, when the purpose of the statistician is to test a hypothesis. Since c_1 and c_2 are costs for doing different things one cannot come up with a simple number which can be called the asymptotic efficiency. Instead, we propose, in Section 5, a cost ratio based on Bayesian considerations. This Asymptotic Bayes Cost Ratio (ABCR(I,II)) is the limiting ratio of the two costs of sampling which acheive equal Bayes risks when the best available testing procedure and best sample size or stopping time are used in each case. Thus, when the actual cost ratio c_1/c_2 is less than ABCR(I,II), scheme I is more profitable and when c_1/c_2 exceeds ABCR(I,II), scheme II is more profitable.

It turns out that the scheme that produces more expected number of observations for the same cost is not necessarily the more profitable scheme. Properties other than the mean of the occurrence of the observations play a role in determining the more profitable scheme.

The calculation of efficiency of tests leads quite naturally to deviation theory. For example, the computation of Bayes risk efficiency as formulated by Chernoff (1952) or the Bahadur efficiency (Bahadur (1960, (1967), and (1971)) requires results from large deviation theory, while the computation of Bayes risk efficiency as formulated by Rubin and Sethuraman (1965a) and (1965b) requires moderate deviation theory. This is also characteristic of the ABCR(I,II). In Section 4, a large deviation result for sequences of random variables with random indices, required in the computation of ABCR(I,II), is established.

We conclude the paper with an example. The ABCR(I,II) is computed when the observations occur according to a Poisson process, and have a common normal or Laplace distribution.

2. Preliminaries.

Let X_1, X_2,... be i.i.d. with distribution P_θ determined by the parameter θ. For $n = 1, 2, \ldots$, let $(X_1,\ldots,X_n) = X_{(n)}$ and let $X_{(0)}$ denote the event that no observations are taken. We assume that the observations occur randomly over time, i.e., the number which occur up until time t, say N_t, is a random variable. For simplicity and convenience we also assume N_t is independent of the observations.

Consider the problem of testing the simple hypothesis H_0: $\theta = \theta_0$ against the simple alternative H_1: $\theta = \theta_1$. To avoid trivialities, we assume that $P_{\theta_0} \neq P_{\theta_1}$ and that P_{θ_0} and P_{θ_1} are not mutually singular, i.e., there does not exist a set A such that $P_{\theta_0}(A) = 1 = P_{\theta_1}(A^c)$. A statistical test of H_0 against H_1 is defined as follows.

Definition. Given a statistic T, (possibly multidimensional), a function ϕ is said to be a test of H_0 against H_1 based on T if

(i) ϕ is a measurable function of T,

(ii) $0 \leq \phi \leq 1$, and

(iii) the test consists of rejecting H_0 with probability $\phi(T)$ when T is observed.

The type I and type II error probabilities are

$$\alpha(\phi) = E_{\theta_0}(\phi(T)) \text{ and } \beta(\phi) = E_{\theta_1}(1-\phi(T)),$$

respectively.

We assume that a loss $k_1(k_2)(0 < k_i < \infty,\ i = 1,2)$ is incurred for a type I (II) error. We place a prior on $\{\theta_0, \theta_1\}$ by assigning masses π and $1 - \pi$ $(0 < \pi < 1)$ to θ_0 and θ_1, respectively. Then the Bayes loss of the test ϕ is given by

$$B(\phi) = \pi k_1 \alpha(\phi) + (1-\pi)k_2\beta(\phi), \tag{2.1}$$

and the Bayes risk, which is the sum of loss and observation cost, is given by

$$M(\phi, c_T) = c_T + B(\phi), \tag{2.2}$$

where c_T denotes the cost of observing the statistic T.

For testing H_0 against H_1, we assume the following two sampling schemes are both available.

Scheme I. Take n observations at a cost c_1 $(0 < c_1 < \infty)$ per observation.

Scheme II. Take N_t (a random variable) observations by observing till some fixed time t at a cost $c_2 (0 < c_2 < \infty)$ per unit time.

The quantities c_1 and c_2 are referred to as the unit costs. It is easily seen that the Bayes risks of a test ϕ_I based on $X_{(n)}$ (i.e. Scheme I) and a test ϕ_{II} based on $X_{(N_t)}$ (i.e. Scheme II) are

$$c_1 n + B(\phi_I) \text{ and } c_2 t + B(\phi_{II}),$$

respectively.

Let f_1 and f_2 denote the density functions of P_{θ_1} and P_{θ_2}, respectively, with respect to some common σ-finite measure ν (which is easy to construct.) For $n > 0$, let

$$(2.3)\quad \phi_n(X_{(n)}) = \begin{cases} 1 & \text{if } \sum_1^n \log[f_1(X_i)/f_2(X_i)] \geq \log[k_1\pi/k_2(1-\pi)] \\ 0 & \text{otherwise} \end{cases}$$

and for $n = 0$, let

$$\phi_0(X_{(0)}) = \begin{cases} 1 & \text{if } k_1\pi \geq k_2(1-\pi) \\ 0 & \text{otherwise.} \end{cases}$$

It is well known that for a fixed number of observations n or for a random number, N_t, of observations obtained by observing till time t, the Bayes loss is minimized by basing the test procedure on the likelihood ratio, for instance see Ferguson ((1967), p. 292). Thus

$$(2.4)\quad \inf\{B(\phi)\colon \phi \text{ a test based on } X_{(n)}\} = B(\phi_n)$$

and

$$(2.5)\quad \inf\{B(\phi)\colon \phi \text{ a test based on } X_{(N_t)}\} = B(\phi_{N_t}).$$

In the next section we compare tests based on the competing sampling schemes I and II, on the basis of their Bayes risks. In each case we choose the best available test and the optimal sample size or the optimal stopping time. In view of (2.4) and (2.5), we can restrict our attention exclusively to likelihood ratio tests.

3. Asymptotic Bayes Cost Ratio.

Let $\phi_n (n=0,1,\ldots)$ be as defined in (2.3). Let

$$B_I(c_1) = \inf_n\{c_1 n + B(\phi_n)\}$$

and

$$B_{II}(c_2) = \inf_t\{c_2 t + B(\phi_{N_t})\}.$$

The quantities $B_I(c_1)$ and $B_{II}(c_2)$ correspond, respectively, to the risks involved when the sample size n for Scheme I and the stopping time t for Scheme II are chosen optimally. In typical cases, $B_I(c)$ and $B_{II}(c)$ both tend to 0 as c tends to 0. When this is the case, we define the Bayes risk efficiency as follows.

Definition. Let $c_1(\cdot)$ be a function of c_2 such that $B_I(c) \le B_{II}(c_2)$ for $0 < c < c_1(c_2)$ and $B_I(c_1(c_2)) \ge B_{II}(c_2)$. Then the asymptotic Bayes cost ratio (ABCR) of Scheme II relative to Scheme I is

$$\mathrm{ABCR(I,II)} = \lim_{c_2 \to 0} c_1(c_2)/c_2$$

when this limit exists.

To put the above definition into more applicable terms, the Bayes risk criterion suggests that it is more profitable for the statistician to use Scheme I when the ratio of the unit costs $c_1/c_2 \le \mathrm{ABCR(I,II)}$ and to use

Scheme II otherwise. Of course, we are assuming that the unit costs are both relatively small.

When both c_1 and c_2 are small, the statistician is likely to observe a large number of observations since other costs remain fixed. One might, as a first guess, say that the sampling scheme which yields more expected number of observations for the same sampling cost is more profitable. For instance, suppose under Scheme II that $E(N_t/t) \to \mu$ as $t \to \infty$, where μ is some fixed constant. If the statistician observes until time t (t large), he will observe approximately μt observations at a sampling cost of tc_2. Under Scheme I, he can observe tc_2/c_1 observations for the same sampling cost. Thus, one might guess that Scheme II would be more profitable than Scheme I if $\mu t > tc_2/c_1$, i.e. if $c_1/c_2 > 1/\mu$. This may not be always correct. We show in Theorem 5.1 that distribution properties of N_t other than the behavior of $E(N_t)/t$ as $t \to \infty$ are involved. In fact, Theorem 5.1 states that under certain conditions,

$$ABCR(I,II) = -\rho/J(-\rho),$$

where ρ depends on the distribution of X_1 through its m.g.f. and J depends on the distribution of N_t through its m.g.f.. However, from Lemma 5.1, wherein it is shown that $ABCR(I,II) \geq 1/\mu$, it follows that Scheme I is more profitable than Scheme II if $c_1/c_2 \leq 1/\mu$.

4. Large deviations.

Let $\{T_n\}$ be a sequence of random variables, defined on some probability space $(\mathcal{X},\mathcal{B},P)$, for which $T_n \to \mu$ in probability as $n \to \infty$. Let A be a Borel measurable set such that the closure of A does not contain μ. Then $P(T_n \in A) \to 0$

Scheme II otherwise. Of course, we are assuming that the unit costs are both relatively small.

When both c_1 and c_2 are small, the statistician is likely to observe a large number of observations since other costs remain fixed. One might, as a first guess, say that the sampling scheme which yields more expected number of observations for the same sampling cost is more profitable. For instance, suppose under Scheme II that $E(N_t/t) \to \mu$ as $t \to \infty$, where μ is some fixed constant. If the statistician observes until time t (t large), he will observe approximately μt observations at a sampling cost of tc_2. Under Scheme I, he can observe tc_2/c_1 observations for the same sampling cost. Thus, one might guess that Scheme II would be more profitable than Scheme I if $\mu t > tc_2/c_1$, i.e. if $c_1/c_2 > 1/\mu$. This may not be always correct. We show in Theorem 5.1 that distribution properties of N_t other than the behavior of $E(N_t)/t$ as $t \to \infty$ are involved. In fact, Theorem 5.1 states that under certain conditions,

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as $n \to \infty$. In typical cases $P(T_n \in A) \to 0$ at an exponential rate, i.e., there exists a positive number I_A such that

(4.1) $$\lim_n \frac{1}{n} \log P(T_n \in A) = -I_A.$$

When (4.1) holds, we say A is a <u>large deviation event</u> and call I_A the <u>index of large deviation</u> of $\{T_n \in A\}$ or of T_n if the set A is otherwise understood.

The earliest research in the theory of large deviations dealt primarily with sums of independent random variables. Most notable is the following theorem due to Chernoff.

<u>Theorem 4.1</u>. (Chernoff (1952), p. 494). Let $Y_1, Y_2, \ldots$ be a sequence of i.i.d. random variables with mean $\mu < \infty$. (μ may equal $-\infty$). Then for $T_n = \sum_1^n Y_i/n$,

$$\lim_n \frac{1}{n} \log P(T_n \geq a) = -I_{[a,\infty)},$$

where

$$-I_{[a,\infty)} = \inf\{-\lambda a + \log E(e^{\lambda Y_1}): \lambda \geq 0\}.$$

As a consequence of the above theorem, Chernoff was able to determine the exponential rate at which $B(\phi_n)$ goes to zero. This is given in the following corollary.

Let

(4.2) $$-\rho = \log \inf\{g(\lambda): 0 < \lambda < 1\},$$

where

$$g(\lambda) = \int [f_1(x)]^\lambda [f_0(x)]^{1-\lambda} d\nu(x).$$

Then,

Corollary 4.2. Let $B(\phi_n)$ be as defined in (2.4). Then,

$$\lim_n \frac{1}{n} \log B(\phi_n) = -\rho. \tag{4.3}$$

Remark. Under our assumption that P_o and P_1 are not mutually singular, it should be noted that $B(\phi_n) \to 0$ as $n \to \infty$ neither too fast nor too slow; i.e., $0 < \rho < \infty$.

To suitably approximate the Bayes risk for Scheme II, we need a result analogous to Corollary 4.2 for $B(\phi_{N_t})$. This will follow from Corollary 4.4 below which is a consequence of Theorem 4.3.

Theorem 4.3. Let $\{a_n\}$ be a sequence of non-negative integers such that

$$\lim_n (1/n) \log a_n = -I, \tag{4.4}$$

where $0 < I < \infty$. Let $\psi_t(\lambda) = E(e^{\lambda N_t})$ and let

$$\lim_t (1/t) \log \psi_t(\lambda) = J(\lambda) \tag{4.5}$$

exist and be continuous and strictly increasing in a neighborhood of $-I$. Let $h(t) = \sum_0^\infty a_n P(N_t = n)$. Then

$$\lim_t (1/t) \log h(t) = J(-I). \tag{4.6}$$

Proof. Let $\varepsilon > 0$. From (4.4),

$$e^{(-I-\varepsilon)k} \le a_k \le e^{(-I+\varepsilon)k}$$

for all $k \ge k(\varepsilon)$. Thus

$$(4.7)\qquad \psi_t(-I-\varepsilon) - \sum_{1}^{k(\varepsilon)} e^{m(-I-\varepsilon)} P(N_t = m) \le h(t) \le \psi_t(-I+\varepsilon) + a(\varepsilon)\, P(N_t \le k(\varepsilon)),$$

where $a(\varepsilon) = \max\{a_k\colon\ 0 \le k \le k(\varepsilon)\}$. Assume that $I > 0$ and choose ε such that $I - \varepsilon > 0$. From a Chebycheff inequality for $P\{N_t \le k(\varepsilon)\}$,

$$h(t) \le \psi_t(-I+\varepsilon)[1 + a(\varepsilon)e^{k(\varepsilon)(I-\varepsilon)}]$$

and

$$\overline{\lim_t}\ (1/t) \log h(t) \le J(-I+\varepsilon).$$

Since J is continuous at $-I$, we can allow ε to decrease to 0 and obtain

$$(4.8)\qquad \overline{\lim_t}\ (1/t) \log h(t) \le J(-I).$$

If $I = 0$, (4.8) is immediate since $J(0) = 0$. Now, choose ε_1 to satisfy $0 < \varepsilon_1 < \varepsilon$ and $J(-I-\varepsilon_1) - J(-I-\varepsilon) = 3\alpha > 0$. From (4.5), there is a t_o such that for $t \ge t_o$

$$e^{t[J(-I-\varepsilon_1)-\alpha]} \le \psi_t(-I-\varepsilon_1) \le \psi_t(-I-\varepsilon) \le e^{t[J(-I-\varepsilon)+\alpha]}.$$

Thus, for $t \ge t_o$, it follows that

$$\begin{aligned}
h(t) &\ge \psi_t(-I-\varepsilon_1) - \sum_{m=0}^{k(\varepsilon_1)} e^{m(-I-\varepsilon_1)} P(N_t = m) \\
&\ge \psi_t(-I-\varepsilon_1) - e^{k(\varepsilon_1)(\varepsilon-\varepsilon_1)} \sum_{m=0}^{k(\varepsilon)} e^{m(-I-\varepsilon)} P(N_t = m) \\
&\ge \psi_t(-I-\varepsilon_1) - e^{k(\varepsilon_1)(\varepsilon-\varepsilon_1)} \psi_t(I-\varepsilon) \\
&\ge e^{t[J(-I-\varepsilon_1)-\alpha]} - e^{k(\varepsilon_1)(\varepsilon-\varepsilon_1) + t[J(-I-\varepsilon)+\alpha]} \\
&\ge e^{t[J(-I-\varepsilon_1)-\alpha]}[1 + o(1)]
\end{aligned}$$

as $t \to \infty$, thus establishing

$$\underline{\lim_{t}}\ (1/t) \log h(t) \geq J(-I-\varepsilon_1)-\alpha.$$

Since J is continuous at $-I$, we may allow $\varepsilon \to 0$ which entails $\varepsilon_1 \to 0$ and $\alpha \to 0$ and we therefore obtain

$$\underline{\lim_{t}}\ (1/t) \log h(t) \geq J(-I). \tag{4.9}$$

Relations (4.8) and (4.9) established Theorem 4.2. □

<u>Corollary 4.4</u>. Let ρ be as defined in (4.2). Let $J(\lambda)$ as defined in (4.5) exist, be continuous and strictly increasing in a neighborhood of $-\rho$. Then

$$\lim_{t}\ (1/t) \log B(\phi_{N_t}) = J(-\rho). \tag{4.10}$$

<u>Proof</u>. Since it was assumed that N_t is independent of $X_1, X_2, \ldots$, we have

$$B(\phi(N_t)) = \sum_{m=0}^{\infty} B(\phi_m)\ P(N_t = m).$$

Corollary 4.4 follows immediately from Theorem 4.2. □

<u>Remark</u>. Let $\{N_t, t \geq 0\}$ be a non homogeneous Poisson process with intensity function $\mu(t)$ satisfying

$$M(t)/t \to \mu, \text{ where } M(t) = \int_0^t \mu(s)ds. \tag{4.11}$$

Then,

$$\psi_t(\lambda) = E(e^{\lambda N_t}) = e^{M(t)(e^{\lambda}-1)},$$

and condition (4.6) of Theorem 4.2 is satisfied, with

$$J(\lambda) = (e^{\lambda}-1)\mu. \tag{4.12}$$

5. General results on ABCR(I,II).

In the remainder of this paper, the following assumption is made concerning the process $\{N_t: t > 0\}$.

Assumption A. Let ρ be as defined in (4.2). There is a function J which is continuous, strictly increasing and satisfies (4.5) in some neighborhood of $-\rho$.

Theorem 5.1. Let assumption A hold. Then

$$\text{ABCR(I,II)} = -\rho/J(-\rho). \tag{5.1}$$

Proof. The theorem follows immediately from the asymptotic rates of $B_I(c)$ $B_{II}(c)$ in (5.2) and (5.3) below, which we proceed to establish:

$$B_I(c) \sim -c \log c/\rho, \text{ and} \tag{5.2}$$

$$B_{II}(c) \sim c \log c/J(-\rho), \tag{5.3}$$

where $f(c) \sim g(c)$ denotes $f(c)/g(c) \to 1$ as $c \to 0$. We will prove only (5.3). The proof of (5.2) is analogous and will be omitted.

Since, $0 < \rho < \infty$ by the Remark following Corollary 4.2, it follows that $0 < -J(-\rho) < \infty$. Choose ε such that $0 < \varepsilon < -J(-\rho)$. Recall that the Bayes risk for Scheme II with unit cost c and observing till time t is

$$M(\phi_{N_t}, ct) = ct + B(\phi_{N_t}).$$

Thus, it follows from (4.10) of Corollary 4.4 that there exists a t_o independent of c such that for $t \ge t_o$

$$\underline{M}_{\varepsilon,c}(t) = e^{t[J(-\rho)-\varepsilon]} + ct$$

$$\le M(\phi_{N_t}, ct)$$

$$\le e^{t[J(-\rho)+\varepsilon]} + ct = \overline{M}_{\varepsilon,c}(t).$$

By differentiating $\underline{M}_{\varepsilon,c}(\cdot)$ and $\overline{M}_{\varepsilon,c}(\cdot)$, it is seen that $\underline{M}_{\varepsilon,c}(\cdot)$ and $\overline{M}_{\varepsilon,c}(\cdot)$ attain their minimums at

$$\underline{t}_c = -[J(-\rho)-\varepsilon]^{-1} \log[-J(-\rho)-\varepsilon)c^{-1}]$$

and

$$\overline{t}_c = -[J(-\rho)+\varepsilon]^{-1} \log[-J(-\rho)+\varepsilon)c^{-1}],$$

respectively. Since $\underline{t}_c$ and $\overline{t}_c$ tend to infinity as $c \to 0$, it follows that

(5.4) $$\underline{M}_{\varepsilon,c}(\underline{t}_c) \le \inf\{M(\phi_{N_t}, ct): t \ge t_o\} \le \overline{M}_{\varepsilon,c}(\overline{t}_c),$$

for all sufficiently small c. As $c \to 0$, both the extreme terms of (5.4) tend to zero. Thus

(5.5) $$\inf\{M(\phi_{N_t}, ct): t \ge t_o\} \to 0 \text{ as } c \to 0.$$

Since $0 < J(-\rho) < \infty$, it follows from (4.10) that $B(\phi_{N_t}) > 0$ for all large t. Since $B(\phi_{N_t})$ is monotonic decreasing in t we can conclude that

$$\inf\{M(\phi_{N_t}, ct): t < t_o\} \ge B(\phi_{N_{t_o}}) > 0$$

and, from (5.5), also that

(5.6) $$\inf\{M(\phi_{N_t}, ct) : t \geq t_o\} = B_{II}(c)$$

if c is small. Since

$$\lim_{\varepsilon} \underline{\lim}_{c} \underline{M}_{\varepsilon,c}(\underline{t}_c)/c \log c = [J(-\rho)]^{-1}$$

and

$$\lim_{\varepsilon} \underline{\lim}_{c} \underline{M}_{\varepsilon,c}(\overline{t}_c)/c \log c = [J(-\rho)]^{-1},$$

it follows from (5.4) and (5.6) that $B_{II}(c) \sim c \log c/J(-\rho)$. This completes the proof of Theorem 5.1. □

Remark. Let $t_c \sim (\log c)/J(-\rho)$. Then it will follow from the proof above and (4.10) that

$$M(\phi_{N_{t_c}}, ct_c) \sim B_{II}(c).$$

Thus for Scheme II with unit cost c, t_c is an asymptotically optimal stopping time. It follows similarly that $n_c \sim \log c/-\rho$ is an asymptotically optimal sample size for Scheme I with unit cost c.

The following two lemmas show the relationship of the ABCR(I,II) to the $E(N_t/t)$ and N_t/t as $t \to \infty$.

Lemmas 5.2. The

$$\text{ABCR(I,II)} \geq 1/\underline{\lim}_{t}\, E(N_t/t).$$

In fact, the above inequality is strict when J is strictly convex.

Proof. Let $\varepsilon > 0$. Since $\log \psi_t(\lambda)$ is the cumulant generating function of N_t, it is convex. Hence, for sufficiently small ε,

$$\frac{1}{-\rho t} \log \psi_t(-\rho) = \frac{1}{-\rho t} [\log \psi_t(-\rho) - \log \psi_t(0)]$$

$$\leq \frac{1}{(-\rho+\varepsilon)t} [\log \psi_t(-\rho+\varepsilon) - \log \psi_t(0)]$$

$$\leq \frac{1}{t} \frac{d}{d\lambda} \log \psi_t(\lambda)\Big|_{\lambda=0} = E(N_t/t).$$

The lemma follows by taking lim inf's in the above and by observing that, when $J(\lambda)$ is strictly convex, $J(-\rho)/-\rho < J(-\rho+\varepsilon)/(-\rho+\varepsilon)$. □

<u>Lemma 5.3</u>. If $N_t/t \to \mu$ in probability as $t \to \infty$, then

$$ABCR(I,II) \geq 1/\mu \geq 1/\underline{\lim}\, E(N_t/t).$$

<u>Proof</u>. The right hand inequality follows readily from Fatou's lemma. By Jensen's inequality, for $t \geq 1$,

$$\frac{1}{t} \log \psi_t(\lambda) \geq \log E(e^{\lambda N_t/t}).$$

By another application of Fatou's lemma

$$\lim_t t^{-1} \log \psi_t(\lambda) = J(\lambda) \geq \lambda\mu.$$

Thus $-\rho/J(-\rho) \geq 1/\mu$. This completes the proof of Lemma 5.3. □

<u>Example</u>. Let $X_1, X_2, \ldots$ be i.i.d. with mean θ and variance σ^2. We now compute the ABCR(I,II) for testing H_0: $\theta = 0$ against H_1: $\theta = \theta_1$ when the observations occur according to a Poisson process $\{N_t: t > 0\}$ with mean rate μ and have a common normal or Laplace distribution. Then the ABCR(I,II) is

$$ABCR(I,II) = -\rho/J(-\rho),$$

where, from (4.2),

$$\rho = \theta_1^2/(8\sigma^2) \text{ if the X's are normal, and}$$

$$\rho = -\log\,(1 + \theta_1/(\sigma\sqrt{2})) + \theta_1/(\sigma\sqrt{2})$$

if the X's are Laplace, and

$$J(\lambda) = \mu(e^{\lambda} - 1).$$

The table below lists μ ABCR(I,II) for various values of θ_1/σ. One interprets the table as follows: Scheme II is more or less profitable than Scheme I according to whether or not $\mu c_1/c_2$ is greater than the appropriate entry in the table. One sees immediately from the table that as the alternative hypothesis recedes from the null hypothesis the unit cost c_2 necessary for Scheme II to be as profitable as Scheme I with a fixed unit cost c_1 must decrease. Also, as is expected from Lemma 5.2, μ ABCR ≥ 1.

Values of μ ABCR(I,II), when under Scheme II observations according to a Poisson process with parameter μ and when testing for

θ/σ	$N(0,\sigma)$ versus $N(\theta,\sigma)$*	$L(0,\sigma)$ versus $L(\theta,\sigma)$*
.125	1.001	1.002
.250	1.004	1.007
.375	1.009	1.015
.5	1.016	1.026
.625	1.025	1.038
.75	1.036	1.053
.875	1.049	1.070
1	1.064	1.089
1.5	1.147	1.178
2	1.271	1.290
2.5	1.440	1.421
3	1.666	1.571
3.5	1.954	1.738
4	2.313	1.921
4.5	2.750	2.119
5	3.269	2.332

*$N(\theta,\sigma)$ = Normal distribution with mean θ and variance σ^2.
$L(\theta,\sigma)$ = Laplace distribution with mean θ and variance σ^2.

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20. ABSTRACT

A large deviation result is established for sequences of random variables with random indices. This result is used to compare two different sampling schemes when testing for a simple hypothesis versus a simple alternative. It turns out that the sampling scheme yielding more expected number of observations for the same sampling cost is not necessarily the more profitable procedure. Properties other than the mean of the distribution of the occurrence of the observations play a role in determining the more profitable sampling rule.